

Final Exam

Answer all 10 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. (8 points) Rewrite the following statement using symbolic logic, then negate it.

$$A \cap B \subseteq C$$

You may use the symbols $\in, \notin, =, \neq, \wedge, \vee, \implies, \iff, \forall$ and \exists in your answer, but not $\subseteq, \not\subseteq, \cap, \cup$, or \sim .

2. Negate the following statements. Say which one is true: the original or the negation.

(a) (4 points) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x^2 = y^2) \implies (x = y)$

(b) (4 points) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x = \frac{1}{y}$

3. (10 points) Suppose $a, b \in \mathbb{Z}$. Prove the following statement.

$(a - 3)b^2$ is even if and only if a is odd or b is even.

4. (10 points) Prove the following statement.

If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

5. (10 points) Prove that $\sqrt{6}$ is irrational.

6. (10 points) Prove the following statement.

If A and B are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

7. (10 points) Let $r \in \mathbb{R} - \{0, 1\}$. Prove by induction that for every $n \in \mathbb{N}$,

$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r - 1}.$$

8. Suppose $A = \{1, 2, 3\}$.

(a) (4 points) How many different equivalence relations on A exist? *Hint: draw diagrams.*

(b) (4 points) How many different relations on A exist? *Hint: do not draw diagrams.*

9. Let R be the relation on \mathbb{Z} given by

$$x R y \iff 3 \mid (x^2 - y^2).$$

(a) (10 points) Prove that R is an equivalence relation on \mathbb{Z} .

(b) (4 points) Find three distinct elements that belong to $[1]$.

10. Consider the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$f(n) = \begin{cases} 4n + 1 & \text{if } n \text{ is even} \\ 3n - 2 & \text{if } n \text{ is odd} \end{cases}$$

(a) (6 points) Determine explicitly $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.

(b) (6 points) Is f injective? Surjective? Bijective? Explain.