Final Exam

Answer all 10 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. (8 points) Rewrite the following statement using symbolic logic, then negate it.

$$A \cap B \subseteq C$$

You may use the symbols $\in, \notin, =, \neq, \wedge, \vee, \implies, \iff, \forall$ and $\exists$ in your answer, but not $\subseteq, \not\subseteq, \cap, \cup$, or $\sim$.

Solution: Original: $\forall x,\ (x \in A \wedge x \in B) \implies x \in C$
Negation: $\exists x,\ x \in A \wedge x \in B \wedge x \notin C$

2. Negate the following statements. Say which one is true: the original or the negation.

(a) (4 points) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ (x^2 = y^2) \implies (x = y)$

Solution: $\exists x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x^2 = y^2) \wedge (x \neq y)$. The negation is true (e.g. $x = 1, y = -1$).

(b) (4 points) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ x = \frac{1}{y}$

Solution: $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ x \neq \frac{1}{y}$. The negation is true (e.g. $x = 0$).

3. (10 points) Suppose $a, b \in \mathbb{Z}$. Prove the following statement.
$(a-3)b^2$ is even if and only if a is odd or b is even.

Solution: First we prove that if $(a-3)b^2$ is even, then a is odd or b is even. We do so by proving the contrapositive statement: if a is even and b is odd, then $(a-3)b^2$ is odd. In brief, a is even implies $(a-3)$ is odd, and b is odd implies b^3 is odd. Therefore $(a-3)b^2$ is the product of two odd numbers and thus odd. In detail, let $a = 2m$ and $b = 2n+1$, $a, n \in \mathbb{Z}$. Then

$$\begin{aligned}(a-3)b^2 &= (2m-3)(2n+1)^2 \\ &= (2m-3)(4n^2+4n+1) \\ &= 8mn^2 + 8mn + 2m - 12n^2 - 12n - 3 \\ &= 2(4mn^2 + 4mn + m - 6n^2 - 6 - 2) + 1.\end{aligned}$$

Since $4mn^2 + 4mn + m - 6n^2 - 6 - 2 \in \mathbb{Z}$, this shows that $(a-3)b^2$ is odd.

Next we show that if a is odd or b is even, then $(a-3)b^2$ is even. We do this directly by cases.

- Case 1: If a is odd, let $a = 2n+1$ $n \in \mathbb{Z}$. Then

$$\begin{aligned}(a-3)b^2 &= (2n+1-3)b^2 \\ &= 2(n-1)b^2.\end{aligned}$$

Since $(n-1)b^2 \in \mathbb{Z}$, this shows that $(a-3)b^2$ is even.

- Case 2: If b is even, let $b = 2n$, $n \in \mathbb{Z}$. Then

$$\begin{aligned}(a-3)b^2 &= (a-3)(2n)^2 \\ &= 2\big(2n^2(a-3)\big).\end{aligned}$$

  Since $2n^2(a-3) \in \mathbb{Z}$, this shows that $(a-3)b^2$ is even.

This completes the proof.

4. (10 points) Prove the following statement.
If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

Solution: Assume $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then $a - b = nj$ and $c - d = nk$ for some $j, k \in \mathbb{Z}$. In other words, $a = b + nj$ and $c = d + nk$. Now

$$\begin{aligned}ac - bd &= (b+nj)(d+nk) - bd \\ &= bd + bnk + dnj + n^2jk - bd \\ &= n(bk + dj + njk)\end{aligned}$$

Since $bk + dj + njk \in \mathbb{Z}$, this shows that $n \mid (ac - bd)$ and thus $ac \equiv bd \pmod{n}$.

5. (10 points) Prove that $\sqrt{6}$ is irrational.

Solution: Assume for the sake of contradiction that $\sqrt{6}$ is rational. That is, assume $\sqrt{6} = a/b$ for some $a, b \in \mathbb{Z}$. Without loss of generality, assume a/b is reduced. In particular, assume <u>a and b are not both even</u>. Now

$$\sqrt{6} = \frac{a}{b} \implies a^2 = 6b^2 = 2(3b^2).$$

Since $3b^2 \in \mathbb{Z}$, we see that a^2 is even and thus <u>a is even</u>. Let $a = 2n$, $n \in \mathbb{Z}$. Then

$$a^2 = 2(3b^2) \implies (2n)^2 = 2(3b^2) \implies 3b^2 = 2n^2.$$

Since $n^2 \in \mathbb{Z}$, we see that $3b^2$ is even and thus <u>b is even</u>. Since our assumption that $\sqrt{6}$ is rational has led to a contradiction, it follows that $\sqrt{6}$ is irrational.

6. (10 points) Prove the folowing statement.
If A and B are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

Solution: We can prove this using the method of double inclusion. First we show that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$. Assume $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \subseteq A$ and $X \subseteq B$. Therefore $X \subseteq A \cap B$, and so $X \in \mathcal{P}(A \cap B)$.

Next we show that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. Assume $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$, which means that $X \subseteq A$ and $X \subseteq B$. Therefore $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, and so $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Since $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, it follows that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

Alternatively, we can prove the statement using only definitions and logical equivalences.

$$\begin{aligned}\mathcal{P}(A)\cap\mathcal{P}(B) &= \{X \;:\; X\subseteq A\}\cap\{X \;:\; X\subseteq B\}\\ &= \{X \;:\; X\subseteq A \wedge X\subseteq B\}\\ &= \{X \;:\; X\subseteq A\cap B\}\\ &= \mathcal{P}(A\cap B).\end{aligned}$$

7. (10 points) Let $r\in\mathbb{R}-\{0,1\}$. Prove by induction that for every $n\in\mathbb{N}$,

$$\sum_{i=0}^{n-1} r^i = \frac{r^n-1}{r-1}.$$

Solution: First we verify that the statement is true of $n=1$.

$$\sum_{i=0}^{1-1} r^i = r^0 = 1 = \frac{r^1-1}{r-1} \quad \checkmark$$

Now we assume that the statement is true when $n=k$ for some some $k\in\mathbb{N}$, and show that this implies that the statement is also true for $n=k+1$. We have

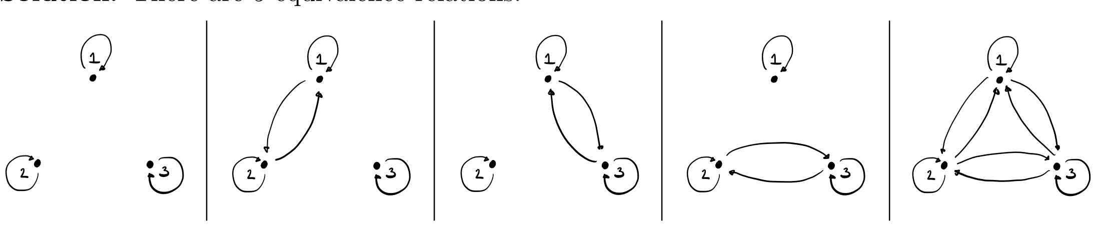
$$\begin{aligned}\sum_{i=0}^{(k+1)-1} r^i &= \sum_{i=0}^{k-1} r^i + r^k\\ &= \frac{r^k-1}{r-1}+r^k\\ &= \frac{r^k-1+r^k(r-1)}{r-1}\\ &= \frac{r^{k+1}-1}{r-1}.\end{aligned}$$

This proves the statement by induction.

8. Suppose $A=\{1,2,3\}$.

   (a) (4 points) How many different equivalence relations on A exist? *Hint: draw diagrams.*

   Solution: There are 5 equivalence relations.



   (b) (4 points) How many different relations on A exist? *Hint: do not draw diagrams.*

Solution: By definition, a relation on A is a subset of the cartesian product $A \times A$. Note that $|A \times A| = |A| \times |A| = 3 \times 3 = 9$. Alternatively, we can see that $A \times A$ contains 9 elements by listing all of its elements.

$$A \times A = \left\{ \begin{array}{ccc} (1,1), & (1,2), & (1,3), \\ (2,1), & (2,2), & (2,3), \\ (3,1), & (3,2), & (3,3) \end{array} \right\}$$

Since $A \times A$ contains 9 elements, it follows that $A \times A$ has $2^9 = 512$ subsets. Thus there exist 512 different relations on A.

9. Let R be the relation on $\mathbb{Z}$ given by

$$x \; R \; y \iff 3 \mid (x^2 - y^2).$$

(a) (10 points) Prove that R is an equivalence relation on $\mathbb{Z}$.

Solution: We must show that R is reflexive, symmetric, and transitive.

- Reflexive: For any $x \in R$, $x^2 - x^2 = 0 = 3 \cdot 0$ and so $3 \mid (x^2 - x^2)$. Therefore xRx.
- Symmetric: Assume $x, y \in \mathbb{R}$ and xRy. Then $3 \mid (x^y - y^2)$, and so $(x^2 - y^2) = 3n$, for some $n \in \mathbb{Z}$. It follows that $(y^2 - x^2) = 3(-n)$, and since $-n \in \mathbb{Z}$, this shows that $3 \mid (y^2 - x^2)$. Therefore yRx.
- Transitive: Assume $x, y, z \in \mathbb{R}$ and xRy and yRz. Since xRy, we have $(x^2 - y^2) = 3m$, for some $m \in \mathbb{Z}$. Similarly, since yRz, we have $(y^2 - z^2) = 3n$ for some $n \in \mathbb{Z}$. Now

$$x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2) = 3m + 3n = 3(m + n).$$

Since $m + n \in \mathbb{Z}$, it follows that $3 \mid (x^2 - z^2)$ and so xRz.

Since R is reflexive, symmetric, and transitive, it follows by definition that R is an equivalence relation on A.

(b) (4 points) Find three distinct elements that belong to [1].

Solution: There are many, e.g. 1, 2, and 4.

$$\begin{aligned} [1] &= \{x \in \mathbb{Z} \;:\; xR1\} \\ &= \{x \in \mathbb{Z} \;:\; 3 \mid (x^2 - 1^2)\} \\ &= \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 11, \pm 14, \pm 16, \ldots\} \end{aligned}$$

10. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} 4n + 1 & \text{if } n \text{ is even} \\ 3n - 2 & \text{if } n \text{ is odd} \end{cases}$$

(a) (6 points) Determine explicitly $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.

Solution: To find $f^{-1}(\{0\})$, we seek either an even integer m such that $4m + 1 = 0$ or an odd integer n such that $3n - 2 = 0$. However, no such integers exist. Therefore

$$f^{-1}(\{0\}) = \emptyset.$$

To find $f^{-1}(\{1\})$, we seek either an even integer m such that $4m+1=1$ or an odd integer n such that $3n-2=1$. Solutions are $m=0$ and $n=1$. Therefore

$$f^{-1}(\{1\}) = \{0, 1\}.$$

(b) (6 points) Is f injective? Surjective? Bijective? Explain.

Solution: None of the above. The function f is not injective because there exist two distinct elements 0 and 1 in the domain $\mathbb{Z}$ such that $f(0) = f(1)$. The function f is not surjective because 0 is in the codomain $\mathbb{Z}$, but 0 is not in the range ($f^{-1}(\{0\}) = \emptyset$). Since a bijective fuction is one that is both injecive and surjective, and f is neither, f is not bijective.