Discrete Mathematics, Spring 2023

Practice problems final exam

Solutions

1. (a) $\forall x, x \in A \implies (x \in B \wedge x \notin C)$.
   Negation: $\exists x, x \in A \wedge (x \notin B \vee x \in C)$.
   (b) We do it in several steps:
   $$\forall X, (X \in \mathscr{P}(A) - \mathscr{P}(B)) \implies X \in \mathscr{P}(A-B)$$
   $$\forall X, (X \subseteq A \wedge X \nsubseteq B) \implies X \subseteq A - B$$
   $$\forall X, (\forall y \in X, y \in A \wedge \exists y \in X, y \notin B) \implies \forall y \in X, y \in A \wedge y \notin B.$$
   Negation:
   $$\exists X, (\forall y \in X, y \in A \wedge \exists y \in X, y \notin B) \wedge (\exists y \in X, y \notin A \vee y \in B).$$

2. (a) $\sim R : \forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y > x \wedge \forall z \in \mathbb{R},\ y \neq z^2 + 5z.$
   (b) $\sim R : \exists a \in A,\ \forall b \in B,\ (a \in C \wedge b \notin C) \vee (b \in C \wedge a \notin C).$

3. (a) Suppose $a^2 \mid b \wedge b^3 \mid c$. Then $b = a^2 k \wedge c = b^3 t$, for some $k, t \in \mathbb{Z}$. It follows that $c = b^3 t = (a^2 k)^3 t = a^6 (k^3 t) \implies a^6 \mid c$.
   (b) Since n is odd, $n = 2k+1$, for some $k \in \mathbb{Z}$. Then $n^2 - 1 = (2k+1)^2 = 4k^2 + 4k + 1 - 1 = 4k(k+1)$. We proceed by cases.
   Case 1: k is even. Then $k = 2t$, with $t \in \mathbb{Z}$. Then $4k(k+1) = 8t(2t+1) \implies 8 \mid n^2 - 1$.
   Case 2: k is odd. Then $k = 2h+1$, with $h \in \mathbb{Z}$. Then $4k(k+1) = 4(2h+1)(2h+1+1) = 4(2h+1)(2h+2) = 8(2h+1)(h+1) \implies 8 \mid n^2 - 1$.
   (c) Suppose $14 \mid n$. Then $n = 14k$, with $k \in \mathbb{Z}$. Thus, $n = 2(7k) = 7(2k) \implies 2 \mid n \wedge 7 \mid n$. Conversely, suppose $2 \mid n \wedge 7 \mid n$. Then $n = 2k \wedge n = 7t$, for some $k, t \in \mathbb{Z}$. It follows that $2k = 7t$. Then $7t$ is even, but since 7 is odd, this implies that t is even. So $t = 2s$, $s \in \mathbb{Z}$. Now we have $n = 7(2s) = 14s$. This shows that $14 \mid n$
   (d) The implication $6 \mid n \wedge 10 \mid n \implies 60 \mid n$ is not true. Counterexample: $n = 30$: we have $6 \mid 30 \wedge 10 \mid 30 \wedge 60 \nmid 30$.

4. We use induction.
   Base step: $4 \mid 3^0 - 1 = 0$ is true.
   Inductive step. Suppose $4 \mid 3^{2k} - 1$, for $k \geq 0$. We have $3^{(2(k+1)} - 1 = 3^{2k} \cdot 9 - 1$. Since $4 \mid 3^{2k} - 1$, we can write $3^{2k} = 4t + 1$, with $t \in \mathbb{Z}$. Then
   $$3^{2k} \cdot 9 - 1 = (4t+1)9 - 1 = 36t + 9 - 1 = 36t + 8 = 4(9t+2) \implies 4 \mid 3^{2(k+1)} - 1.$$

5. (a) Reflexive: $\forall x \in \mathbb{Z}, x\,R\,x$ since $x = x$.

   Symmetric: suppose $x, y \in \mathbb{Z}$ and $x\,R\,y$. Then $x = y \vee xy > 0 \Longrightarrow y = x \vee yx > 0 \Longrightarrow y\,R\,x$.

   Transitive: suppose $x, y, z \in \mathbb{Z}$ and $x\,R\,y \wedge y\,R\,z$. Then $x = y \vee xy > 0$ and $y = z \vee yz > 0$. We proceed by cases.

   Case 1. If $x = y$ is true:

   i. if $y = z$ is true, then $x = y = z \Longrightarrow x = z$.
   ii. If $yz > 0$ is true, then $x = y \wedge yz > 0 \Longrightarrow xz > 0$.

   $\Longrightarrow x = z \vee xz > 0 \Longrightarrow x\,R\,z$.

   Case 2. If $xy > 0$ is true:

   i. if $y = z$ is true, then $y = z \wedge xy > 0 \Longrightarrow xz > 0$.
   ii. If $yz > 0$ is true, then $xy > 0 \wedge yz > 0 \Longrightarrow xy^2z > 0 \Longrightarrow xz > 0$.

   $\Longrightarrow xz > 0 \Longrightarrow x = z \vee xz > 0$ is true $\Longrightarrow x\,R\,z$.

   (b) The set of equivalence classes is $\{[x] : x \in \mathbb{Z}\}$. If $x \in \mathbb{Z}$, then $[x] = \{y \in \mathbb{Z} : x\,R\,y\} = \{y \in \mathbb{Z} : x = y \vee xy > 0\}$. In particular, we have $[0] = \{y \in \mathbb{Z} : 0 = y \vee 0 > 0\} = \{0\}$, $[1] = \{y \in \mathbb{Z} : y = 1 \vee y > 0\} = \mathbb{N}$ and $[-1] = \{y \in \mathbb{Z} : y = -1 \vee y < 0\} = \mathbb{Z}^-$, where $\mathbb{Z}^{-1}$ denotes the set of negative integers. Every integer is in exactly one of these three equivalence classes. Therefore there are only three equivalence classes.

6. We have $R^{-1} = \{(y,x) \in A \times A : (x,y) \in R\}$ and $S = R \cap R^{-1} = \{(x,y) \in A \times A : (x,y) \in R \wedge (x,y) \in R^{-1}\} = \{(x,y) \in A \times A : x\,R\,y \wedge y\,R\,x\}$. We now prove that S is an equivalence relation.

   Reflexive: since R is reflexive, we have $x\,R\,x \wedge x\,R\,x \Longrightarrow (x,x) \in S \Longrightarrow S$ is reflexive.

   Symmetric: let $(x,y) \in S$. Then $(x,y) \in R \wedge (y,x) \in R \Longrightarrow x\,R\,y \wedge y\,R\,x \Longrightarrow y;R;x \wedge x\,R\,y \Longrightarrow (y,x) \in S$, which implies that S is symmetric.

   Transitive: suppose $(x,y) \in S \wedge (y,z) \in S$. Thus, $x\,R\,y \wedge y\,R\,x \wedge y\,R\,z \wedge z\,R\,y \Longrightarrow (x\,R\,y \wedge y\,R\,z) \wedge (z\,R\,y \wedge y\,R\,x) \Longrightarrow$ (since R is transitive) $x\,R\,z \wedge z\,R\,x \Longrightarrow (x,z) \in S$, which implies that S is transitive.

7. We first of all write down the partitions of A:

   (a) $\{\{1,2,3\}\}$.
   (b) $\{\{1\},\{2,3\}\}$.
   (c) $\{\{2\},\{1,3\}\}$.
   (d) $\{\{3\},\{1,2\}\}$.
   (e) $\{\{1\},\{2\},\{3\}\}$.

   To each partition it corresponds an equivalence relation.

   (a) $R = \{(1,1),(2,2),(3,3),(1,3),(3,1),(2,3),(3,2),(1,2),(2,1)\}$.
   (b) $R = \{(1,1),(2,3),(3,2),(2,2),(3,3)\}$.
   (c) $R = \{(2,2),(1,3),(3,1),(3,3),(1,1)\}$.
   (d) $R = \{(3,3),(1,2),(2,1),(1,1),(2,2)\}$.
   (e) $R = \{(1,1),(2,2),(3,3)\}$.

8. (a) We have $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$ and $f(f^{-1}(Y)) = \{f(x) : x \in f^{-1}(Y)\}$. Let $b \in f(f^{-1}(Y))$. Then $b = f(a)$, for some $a \in f^{-1}(Y)$. This means $f(a) \in Y$, which implies $b \in Y \Longrightarrow f(f^{-1}(Y)) \subseteq Y$.

(b) We use double inclusion. Let $x \in f^{-1}(Y \cup Z)$. This means $f(x) \in Y \cup Z \Longrightarrow f(x) \in Y \vee f(x) \in Z \Longrightarrow x \in f^{-1}(Y) \vee x \in f^{-1}(Z) \Longrightarrow x \in f^{-1}(Y) \cup f^{-1}(Z)$, which implies $f^{-1}(Y \cup Z) \subseteq f^{-1}(Y) \cup f^{-1}(Z)$. Conversely, suppose $x \in f^{-1}(Y) \cup f^{-1}(Z)$. This implies $x \in f^{-1}(Y) \vee x \in f^{-1}(Z) \Longrightarrow f(x) \in Y \vee f(x) \in Z \Longrightarrow f(x) \in Y \cup Z \Longrightarrow x \in f^{-1}(Y \cup Z)$. Therefore, $f^{-1}(Y) \cup f^{-1}(Z) \subseteq f^{-1}(Y \cup Z)$.

(c) We use double inclusion. Let $x \in f^{-1}(Y \cap Z)$. Then $f(x) \in Y \cap Z \Longrightarrow f(x) \in Y \wedge f(x) \in Z \Longrightarrow x \in f^{-1}(Y) \wedge x \in f^{-1}(Z) \Longrightarrow x \in f^{-1}(Y) \cap f^{-1}(Z)$, which implies $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$. Conversely, let $x \in f^{-1}(Y) \cap f^{-1}(Z)$. Then $x \in f^{-1}(Y) \wedge x \in f^{-1}(Z) \Longrightarrow f(x) \in Y \wedge f(x) \in Z \Longrightarrow f(x) \in Y \cap Z \Longrightarrow x \in f^{-1}(Y \cap Z)$, which implies $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$.

9. Injective: suppose $\phi(x,y) = \phi(x',y')$. This implies $(x+y, x-y) = (x'+y', x'-y')$, i.e.

$$\begin{cases} x+y = x'+y' \\ x-y = x'-y' \end{cases} \Longrightarrow x = x' \wedge y = y' \Longrightarrow (x,y) = (x',y').$$

Surjective: let $(a,b) \in \mathbb{R}^2$. We solve the equation $\phi(x,y) = (a,b)$. We have

$$\begin{cases} x+y = a \\ x-y = b \end{cases} \Longrightarrow \begin{cases} x = \dfrac{a+b}{2} \\ y = \dfrac{a-b}{2} \end{cases}$$

It follows that ϕ is surjective and its inverse is given by $\phi^{-1}(x,y) = \left(\dfrac{x+y}{2}, \dfrac{x-y}{2}\right)$.

10. (a) Injective: let $n, m \in \mathbb{Z}$, and suppose $f(n) = f(m)$. We proceed by cases.
Case 1: $n, m \geq 0$. Then $f(n) = f(m)$ implies $n^2 - 4 = m^2 - 4 \Longrightarrow n^2 = m^2 \Longrightarrow n = m$ (since $n, m \geq 0$).
Case 2: $n, m < 0$. Then $f(n) = f(m)$ implies $\dfrac{3n}{5} = \dfrac{3m}{5} \Longrightarrow n = m$.
Case 3: $n \geq 0$, $m < 0$. Then $f(n) = f(m)$ implies $n^2 - 4 = \dfrac{3m}{5}$. If we set $n = 1$ and $m = -5$, we have $f(1) = f(-5)$. Thus, f is not injective.
Surjective: let $a = \dfrac{1}{2}$. The equation $f(n) = \dfrac{1}{2}$ has no solutions. In fact, if $n \geq 0$, the equation $n^2 - 4 = \dfrac{1}{2}$ has no solutions in $\mathbb{Z}$, and if $n < 0$, the equation $\dfrac{3}{5}n = \dfrac{1}{2}$ also has no solutions in $\mathbb{Z}$.

(b) We want to prove by induction that $\displaystyle\sum_{i=1}^{n}(i^2 - 4) = \frac{2n^3 + 3n^2 - 23n}{6}$, for every $n \in \mathbb{N}$.
Base step: set $n = 1$. We have $-3 = \dfrac{-18}{6} = -3$, which is true.
Inductive step: assume $\displaystyle\sum_{i=1}^{k}(i^2 - 4) = \frac{2k^3 + 3k^2 - 23k}{6}$, for $k \geq 1$. We have

$$\sum_{i=1}^{k+1}(i^2 - 4) = \sum_{i=1}^{k}(i^2 - 4) + (k+1)^2 - 4 = \frac{2k^3 + 3k^2 - 23k}{6} + k^2 + 2k + 1 - 4$$

$$= \ldots = \frac{2k^3 + 9k^2 - 11k - 18}{6} = \text{(check) } \frac{2(k+1)^3 + 3(k+1)^2 - 23(k+1)}{6}.$$

11. We define a function $f : A \to B$ by writing explicitly $f(1), f(2), f(3)$:

   (a) $f(1) = \alpha, f(2) = \alpha, f(3) = \alpha$.
   (b) $f(1) = \alpha, f(2) = \alpha, f(3) = \beta$.
   (c) $f(1) = \alpha, f(2) = \beta, f(3) = \alpha$.
   (d) $f(1) = \alpha, f(2) = \beta, f(3) = \beta$.
   (e) $f(1) = \beta, f(2) = \alpha, f(3) = \alpha$.
   (f) $f(1) = \beta, f(2) = \alpha, f(3) = \beta$.
   (g) $f(1) = \beta, f(2) = \beta, f(3) = \alpha$.
   (h) $f(1) = \beta, f(2) = \beta, f(3) = \beta$.

   None of the functions are injective. All apart from (a) and (g) are surjective.

12. Reflexive: $\forall x \in X,\ x\,R\,x$ since $f(x) = f(x)$.

   Symmetric: $\forall x, y \in X, x\,R\,y \Longrightarrow f(x) = f(y) \Longrightarrow y\,R\,x$.

   Transitive: $\forall x, y, z \in X,\ x\,R\,y \wedge y\,R\,z \Longrightarrow f(x) = f(y) \wedge f(y) = f(z) \Longrightarrow f(x) = f(y) = f(z) \Longrightarrow f(x) = f(z) \Longrightarrow x\,R\,z$.