Homework 4:
2.9 #2, 6
2.10 #2, 4
3.2 #2, 4, 6, 10
3.3 #4, 6, 8, 12
3.4 #12, 14, 16
3.5 #8
3.6 #2, 4
3.7 #2, 10
3.9 #2

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Exercises for Section 2.9

Translate each of the following sentences into symbolic logic.

2. The number x is positive but the number y is not positive.

$$(x>0) \wedge (y \leq 0)$$

6. For every positive number ε there is a positive number M for which $|f(x)-b|<\varepsilon$, whenever $x>M$.

$$\forall \varepsilon>0, \ \exists M>0, \ (x>M) \Rightarrow |f(x)-b|<\varepsilon$$

NOTE: THIS IS ALSO THE SAME AS $\lim_{x\to\infty} f(x) = b$

Exercises for Section 2.10

Negate the following sentences.

2. If x is prime, then $\sqrt{x}$ is not a rational number.

$\hookrightarrow \forall x \in P, \ \sqrt{x} \notin \mathbb{Q}.$

$$\sim(\forall x \in P, \ \sqrt{x} \notin \mathbb{Q}) = \exists x \in P, \ \sim(\sqrt{x} \notin \mathbb{Q})$$

$$= \exists x \in P, \ \sqrt{x} \in \mathbb{Q}$$

THERE EXISTS A PRIME NUMBER WITH A RATIONAL SQUARE ROOT.

4. For every positive number ε, there is a positive number δ such that $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

$\hookrightarrow \forall \varepsilon > 0, \exists \delta > 0, |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$

$\sim(\forall \varepsilon > 0, \exists \delta > 0, |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$

$= \exists \varepsilon > 0, \sim(\exists \delta > 0, |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$

$= \exists \varepsilon > 0, \forall \delta > 0, \sim(|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$

$= \exists \varepsilon > 0, \forall \delta > 0, \sim(\forall x \in \{x : |x-a| < \delta\}, |f(x)-f(a)| < \varepsilon)$

$= \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \{x : |x-a| < \delta\}, \sim(|f(x)-f(a)| < \varepsilon)$

$= \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \{x : |x-a| < \delta\}, |f(x)-f(a)| \geq \varepsilon$

THERE EXISTS A POSITIVE NUMBER ε SUCH THAT FOR ANY POSITIVE NUMBER δ, THERE IS A NUMBER x WITH $|x-a|<\delta$ SUCH THAT $|f(x)-f(a)| \geq \varepsilon$.

Exercises for Section 3.2

2. Airports are identified with 3-letter codes. For example, Richmond, Virginia has the code *RIC*, and Memphis, Tennessee has *MEM*. How many different 3-letter codes are possible?

POSSIBILITIES

$$\underset{\text{1st letter}}{\boxed{26}} \times \underset{\text{2nd letter}}{\boxed{26}} \times \underset{\text{3rd letter}}{\boxed{26}} = 26^3 = 17{,}576$$

4. In ordering coffee you have a choice of regular or decaf; small, medium or large; here or to go. How many different ways are there to order a coffee?

$$\underset{\text{REG/DECAF}}{\boxed{2}} \times \underset{\text{S/M/L}}{\boxed{3}} \times \underset{\text{HERE/TO GO}}{\boxed{2}} = 12$$

6. You toss a coin, then roll a dice, and then draw a card from a 52-card deck. How many different outcomes are there? How many outcomes are there in which the dice lands on ⚂? How many outcomes are there in which the dice lands on an odd number? How many outcomes are there in which the dice lands on an odd number and the card is a King?

$\boxed{2} \times \boxed{6} \times \boxed{52} = 624$

COIN DICE CARD

$\boxed{2} \quad \boxed{1} \quad \boxed{52} = 104$

COIN ⚂ CARD

$\boxed{2} \quad \boxed{3} \quad \boxed{52} = 312$

COIN ⚀ ⚂ OR ⚄ CARD

$\boxed{2} \times \boxed{3} \times \boxed{4} = 24$

COIN ⚀ ⚂ OR ⚄ KING

10. A dice is tossed four times in a row. There are many possible outcomes, such as ⚁⚅⚅⚂, or ⚂⚂⚂⚂. How many different outcomes are possible?

$\boxed{6} \times \boxed{6} \times \boxed{6} \times \boxed{6} = 6^4 = 1296$

1st DICE 2nd DICE 3rd DICE 4th DICE

Exercises for Section 3.3

4. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such lineups are there in which exactly one of the 5 cards is a queen?

● = QUEEN

● = NOT QUEEN

$\underline{4} \times \underline{48} \times \underline{47} \times \underline{46} \times \underline{45} = 18,679,680$

$\underline{48} \times \underline{4} \times \underline{47} \times \underline{46} \times \underline{45} = 18,679,680$

$\underline{48} \times \underline{47} \times \underline{4} \times \underline{46} \times \underline{45} = 18,679,680$

$\underline{48} \times \underline{47} \times \underline{46} \times \underline{4} \times \underline{45} = 18,679,680$

$\underline{48} \times \underline{47} \times \underline{46} \times \underline{45} \times \underline{4} = 18,679,680$

ADD → 93,398,400

6. Consider lists made from the symbols A, B, C, D, E, with repetition allowed.

(a) How many such length-5 lists have at least one letter repeated?

(b) How many such length-6 lists have at least one letter repeated?

(a) LET $\mathcal{U}$ = SET OF LENGTH-5 LISTS WITH REPETITION ALLOWED

X = SET OF LENGTH-5 LISTS WITH AT LEAST ONE LETTER REPEATED

$\overline{X}$ = SET OF LENGTH-5 LISTS WITH 0 LETTERS REPEATED

$$|\mathcal{U}| = \underline{5} \times \underline{5} \times \underline{5} \times \underline{5} \times \underline{5} = 5^5 = 3125$$

$$|\overline{X}| = \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 5! = 120$$

$$|X| = |\mathcal{U}| - |\overline{X}| = 3125 - 120 = 3005$$

(b) SINCE THERE ARE ONLY 5 DISTINCT LETTERS,

ALL LIST OF LENGTH 6 HAVE AT LEAST ONE LETTER REPEATED

LISTS OF LENGTH 6 IS $5^6 = 15{,}625$

8. This problem concerns lists made from the letters $A, B, C, D, E, F, G, H, I, J$.

(a) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin with a vowel?

(b) How many length-5 lists can be made from these letters if repetition is not allowed and the list must begin and end with a vowel?

(c) How many length-5 lists can be made from these letters if repetition is not allowed and the list must contain exactly one A?

(a) $\underline{3} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} = 9{,}072$

(b) $\underline{3} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{2} = 2{,}016$

(c) TOTAL # LISTS WITH NO REPEATS $= \underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} = 30{,}240$

TOTAL # LISTS WITH NO REPEATS AND 0 A'S $= \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{5} = 15{,}120$

TOTAL # LISTS WITH NO REPEATS AND EXACTLY 1 A $= 30{,}240 - 15{,}120 = 15{,}120$

(HALF)

12. Six math books, four physics books and three chemistry books are arranged on a shelf. How many arrangements are possible if all books of the same subject are grouped together?

LET'S BREAK THIS TASK INTO 4 STEPS & COUNT HOW MANY POSSIBLE CHOICES THERE ARE FOR EACH STEP.

(1) ARRANGE 6 MATH BOOKS IN $6!$ WAYS
(2) ARRANGE 4 PHYSICS BOOKS IN $4!$ WAYS
(3) ARRANGE 3 CHEM BOOKS IN $3!$ WAYS

(4) ARRANGE 3 SUBJECTS IN $3!$ WAYS

BY MULTIPLICATION PRINCIPLE, WE MULTIPLY TO GET TOTAL # WAYS TO COMPLETE ALL 4 STEPS

$$6! \times 4! \times 3! \times 3! = 720 \times 24 \times 6 \times 6 = 622{,}080$$

Exercises for Section 3.4

12. You deal 7 cards off of a 52-card deck and line them up in a row. How many possible lineups are there in which no card is a club?

THERE ARE 13 CLUBS, THUS $52 - 13 = 39$ NON-CLUBS.

7-PERMUTATIONS FROM 39 NON-CLUBS IS

$$P(39,7) = \frac{39!}{(39-7)!} = \frac{39!}{32!} = 77{,}519{,}922{,}480$$

14. Five of ten books are arranged on a shelf. In how many ways can this be done?

$$P(10,5) = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 30{,}240$$

16. How many 4-permutations are there of the set $\{A,B,C,D,E,F\}$ if whenever A appears in the permutation, it is followed by E?

A 1^{st}	:	A	E	___	___	: 4×3	$= 12$	SUM 156
A 2^{nd}	:	___	A	E	___	: 4×3	$= 12$	
A 3^{rd}	:	___	___	A	E	: 4×3	$= 12$	
No A	:	___	___	___	___	: $5 \times 4 \times 3 \times 2$	$= 120$	

Exercises for Section 3.5

6. $|\{X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| = 4\}| =$ $\binom{10}{4} = \frac{10!}{4!\,6!} = 210$

10. A department consists of 5 men and 7 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?

$$\underbrace{\binom{5}{3}}_{\substack{\text{STEP 1:} \\ \text{CHOOSE MEN}}} \times \underbrace{\binom{7}{2}}_{\substack{\text{STEP 2:} \\ \text{CHOOSE WOMEN}}} = \frac{5!}{3!\,2!} \times \frac{7!}{2!\,5!} = 10 \times 21 = 210$$

16. How many 6-element subsets of $A = \{0,1,2,3,4,5,6,7,8,9\}$ have exactly three even elements? How many do not have exactly three even elements?

$$\underbrace{\binom{5}{3}}_{\substack{\text{CHOOSE} \\ \text{EVEN}}} \times \underbrace{\binom{5}{3}}_{\substack{\text{CHOOSE} \\ \text{ODD}}} = \frac{5!}{3!\,2!} \times \frac{5!}{3!\,2!} = 10 \times 10 = 100$$

6-ELEMENT SUBSETS THAT DO NOT HAVE EXACTLY 3 EVEN ELEMENTS

= # 6-ELEMENT SUBSETS − # 6 ELEMENT SUBSETS WITH 3 EVEN ELEMENTS

$$= \binom{10}{6} - 100 = 210 - 100 = 110$$

Exercises for Section 3.6

2. Use the binomial theorem to find the coefficient of x^8y^5 in $(x+y)^{13}$.

BINOMIAL THM: $(x+y)^{13} = \sum_{k=0}^{13} \binom{13}{k} \underbrace{x^{13-k} y^{k}}_{= x^8 y^5 \text{ WHEN } k=5}$

$$= \binom{13}{5} = \frac{13!}{5!\,8!} = \frac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 1287$$

4. Use the binomial theorem to find the coefficient of x^6y^3 in $(3x-2y)^9$.

BINOMIAL THM: $(3x-2y)^9 = \sum_{k=0}^{9} \binom{9}{k} (3x)^{9-k} (-2y)^k$

i.e. $(3x+(-2y))^9$

$$= \sum_{k=0}^{9} \underbrace{\binom{9}{k} 3^{9-k} (-2)^k}\ \underbrace{x^{9-k} y^k}_{= x^6y^3 \text{ WHEN } k=3}$$

$$= \binom{9}{3} 3^6 (-2)^3 = \frac{9!}{3!\,6!} \cdot 3^6(-2)^3$$

$$= \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} \cdot 3^6(-2)^3 = (84)(729)(-8)$$

$$= -489{,}888$$

Exercises for Section 3.7

2. How many 4-digit positive integers are there for which there are no repeated digits, or for which there may be repeated digits, but all digits are odd?

LET $\mathcal{U}$ = SET OF 4-DIGIT POS. INTEGERS (<u>NOTE:</u> CANNOT BEGIN WITH 0)
$A \subseteq \mathcal{U}$ WITH NO REPEATED DIGITS.
$B \subseteq \mathcal{U}$ ALL ODD DIGITS.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= (9\cdot 9\cdot 8\cdot 7) + (5\cdot 5\cdot 5\cdot 5) - (5\cdot 4\cdot 3\cdot 2)$$

$$= 4536 + 625 - 120$$

$$= 5041$$

10. How many 6-digit numbers are even or are divisible by 5?

LET $\mathcal{U}$ = SET OF 6-DIGIT NUMBERS (DON'T BEGIN WITH 0)
$E \subseteq \mathcal{U}$ EVEN NUMBERS (END IN 0, 2, 4, 6, OR 8)
$F \subseteq \mathcal{U}$ DIV. BY 5 (END IN 0 OR 5)

$$|E \cup F| = |E| + |F| - |E \cap F|$$

$$= (9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 5) + (9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 2) - (9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 1)$$

$$= 450{,}000 + 180{,}000 - 90{,}000$$

$$= 540{,}000$$

Exercises for Section 3.9

2. You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

IF YOU DEAL 4 ♡s, 4 ◇, 4 ♧, AND 4 ♤ (16 CARDS TOTAL) THEN THE NEXT CARD (REGARDLESS OF SUIT) MUST CAUSE THE PILE TO HAVE 5 CARDS OF THE SAME SUIT.

$\therefore$ 17 CARDS