HOMEWORK 5: CH. 4 # 2, 6, 8, 12, 14

JOHN ADAMSKI
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2. If x is an odd integer, then x^3 is odd.

PROOF. ASSUME $x \in \mathbb{Z}$ IS ODD.

BY DEFINITION, $x = 2a+1$ FOR SOME $a \in \mathbb{Z}$.

THEN $x^3 = (2a+1)^3 = 8a^3 + 12a^2 + 6a + 1 = 2(4a^3 + 6a^2 + 3a) + 1$

SET $c = 4a^3 + 6a^2 + 3a \in \mathbb{Z}$

THEN $x^3 = 2c + 1$.

THEREFORE, x^3 IS ODD. ∎

6. Suppose $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $a \mid c$, then $a \mid (b+c)$.

PROOF. ASSUME $a \mid b$ AND $a \mid c$.

BY DEFINITION, $b = ax$ AND $c = ay$ FOR SOME $x, y \in \mathbb{Z}$.

THEN $b + c = ax + ay = a(x+y)$.

SET $d = x + y \in \mathbb{Z}$.

THEN $b + c = ad$.

THEREFORE $a \mid (b+c)$. ∎

8. Suppose a is an integer. If $5 \mid 2a$, then $5 \mid a$.

PROOF. ASSUME $5 \mid 2a$.

BY DEFINITION, $2a = 5x$ FOR SOME $x \in \mathbb{Z}$.

BY DEFINITION, $5x$ IS EVEN.

SINCE THE PRODUCT OF TWO ODD NUMBERS IS ODD AND 5 IS ODD, IT MUST BE THAT x IS EVEN.

THAT IS, $x = 2n$, $n \in \mathbb{Z}$.

THUS $2a = 5x = 5(2n) = 2(5n)$.

DIVIDING BOTH SIDES OF $2a = 2(5n)$ BY 2 YIELDS $a = 5n$.

THEREFORE, $5 \mid a$. $\blacksquare$

12. If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

PROOF: ASSUME $0 < x < 4$.

WE HAVE $(x-2)^2 \geq 0$. THAT IS, $x^2 - 4x + 4 \geq 0$, i.e. $4 \geq 4x - x^2$.

FACTORING GIVES $4 \geq x(4-x)$.

SINCE $x > 0$ AND $4 - x > 0$ BY ASSUMPTION, DIVIDING BOTH SIDES BY $x(4-x) > 0$ YIELDS

$$\frac{4}{x(4-x)} \geq 1. \quad \blacksquare$$

14. If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd. (Try cases.)

PROOF: WE CONSIDER 2 CASES BASED ON THE PARITY OF n.

CASE 1. ASSUME $n \in \mathbb{Z}$ IS ODD.
BY DEFINITION, $n = 2a + 1$ FOR SOME $a \in \mathbb{Z}$.
THEN

$$\begin{aligned} 5n^2 + 3n + 7 &= 5(2a+1)^2 + 3(2a+1) + 7 \\ &= 20a^2 + 8a + 15 \\ &= 2(10a^2 + 4a + 7) + 1. \end{aligned}$$

SET $c = 10a^2 + 4a + 7 \in \mathbb{Z}$.
THEN $5n^2 + 3n + 7 = 2c + 1$.
THEREFORE, $5n^2 + 3n + 7$ IS ODD.

CASE 2. ASSUME $n \in \mathbb{Z}$ IS EVEN.
BY DEFINITION, $n = 2a$ FOR SOME $a \in \mathbb{Z}$.
THEN

$$\begin{aligned} 5n^2 + 3n + 7 &= 5(2a)^2 + 3(2a) + 7 \\ &= 20a^2 + 6a + 7 \\ &= 2(10a^2 + 3a + 3) + 1. \end{aligned}$$

SET $c = 10a^2 + 3a + 3 \in \mathbb{Z}$.
THEN $5n^2 + 3n + 7 = 2c + 1$.
THEREFORE, $5n^2 + 3n + 7$ IS ODD.

CASE 1 & CASE 2 SHOW THAT FOR ALL $n \in \mathbb{Z}$, $5n^2 + 3n + 7$ IS ODD. $\blacksquare$