HOMEWORK 7 SOLUTIONS - DISCRETE MATH SPRING 2023

JOHN ADAMSKI

`7.4` **Theorem.** *Given an integer a, then $a^2 + 4a + 5$ is odd if and only if a is even.*

Proof. We prove this biconditional statement in two steps. First, we show directly that if a is even then $a^2 + 4a + 5$ is odd. So let us assume that a is even and set $a = 2n$ for some integer n. Then

$$\begin{aligned} a^2 + 4a + 5 &= (2n)^2 + 4(2n) + 5 \\ &= 2(2n^2 + 4n + 2) + 1. \end{aligned}$$

Since $2n^2 + 4n + 2$ is an integer, this shows that $a^2 + 4a + 5$ is odd.

Next, we show that if $a^2 + 4a + 5$ is odd then a is even. To do this we will prove the contrapositive statement: if a is odd then $a^2 + 4a + 5$ is even. So let us assume that a is odd and set $a = 2n + 1$ for some integer n. Then

$$\begin{aligned} a^2 + 4a + 5 &= (2n + 1)^2 + 4(2n + 1) + 5 \\ &= 2(2n^2 + 6n + 5). \end{aligned}$$

Since $2n^2 + 6n + 5$ is an integer, this shows that $a^2 + 4a + 5$ is even. This completes the proof. □

`7.8` **Theorem.** *Suppose $a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{10}$ if and only if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.*

Proof. First, let us assume that $a \equiv b \pmod{10}$ and show directly that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. By definition, $a - b = 10n$ for some integer n, and so

$$a - b = 2(5n) = 5(2n). \tag{1}$$

Since both $5n$ and $2n$ are integers, it follows that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$.

Now let us prove the converse by assuming $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$ then showing that $a \equiv b \pmod{10}$. By definition, we have $a - b = 2x$ and $a - b = 5y$ for some integers x and y. Thus $2x = 5y$ and, in particular, $5y$ is even. Since 5 is odd, it follows that y must be even. That is, $y = 2m$ for some integer m. Therefore

$$a - b = 5(2m) = 10m,$$

and so $a \equiv b \pmod{1-}$. □

`7.16` **Theorem.** *Suppose $a, b \in \mathbb{Z}$. If ab is odd, then $a^2 + b^2$ is even.*

Proof. In order to prove the statement directly, let us assume that ab is odd. It follows that both a and b are odd, for otherwise ab would be even. Set $a = 2n + 1$ and $b = 2m + 1$ for some integers m and n. Then

$$\begin{aligned} a^2 + b^2 &= (2n+1)^2 + (2m+1)^2 \\ &= 2(2n^2 + 2m^2 + 2n + 2m + 1). \end{aligned}$$

Since $2n^2 + 2m^2 + 2n + 2m + 1$ is an integer, this shows that $a^2 + b^2$ is even. $\square$

`7.22` **Theorem.** *If $n \in \mathbb{Z}$, then $4 \mid n^2$ or $4 \mid (n^2 - 1)$.*

Proof. We break this into cases based on the parity of n. If n is even then set $n = 2x$ for some integer x. Then

$$n^2 = (2x)^2 = 4x^2.$$

Since x^2 is an integer, this shows that $4 \mid n^2$.

Now let us consider the case that n is odd. Set $n = 2y + 1$ for some integer y. Then

$$\begin{aligned} n^2 - 1 &= (2y+1)^2 - 1 \\ &= 4(y^2 + y). \end{aligned}$$

Since $y^2 + y$ is an integer, this shows that $4 \mid (n^2 - 1)$. In summary, for any integer n, it is true that $4 \mid n^2$ or $4 \mid (n^2 - 1)$. $\square$

`8.4` **Theorem.** *If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.*

Proof. Suppose $a \in \{x \in \mathbb{Z} : mn \mid x\}$. Then a is an integer and $mn \mid a$. Set $a = (mn)b$ for some integer b. Then

$$a = m(nb) = n(mb).$$

Since both nb and mb are integers, this shows that $m \mid a$ and $n \mid a$. Thus

$$a \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}.$$

This completes the proof. $\square$

`8.6` **Theorem.** *Suppose A, B and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.*

Proof. To prove this directly, let us assume that $A \subseteq B$ and $x \in A - C$. We must show that $x \in B - C$. Since $x \in A - C$, by definition $x \in A$ and $x \notin C$. Since $A \subseteq B$, it follows that $x \in B$. Therefore, $x \in B$ and $x \notin C$. That is, by definition, $x \in B - C$. $\square$

`lemma` **Lemma 1.** *(Distributive Rule) Let P, Q and R be statements. Then*

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R).$$

Proof. The following truth table proves the logical equivalence.

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

∎

8.8 **Theorem.** *If A, B and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Proof. We prove this directly using only definitions and logically equivalent statements. Note that the logical equivalence of lines 2 and 3 follows immediately from the previous lemma.[1]

$$\begin{aligned} A \cup (B \cap C) &= \{x : (x \in A) \vee (x \in B \cap C)\} \\ &= \{x : (x \in A) \vee ((x \in B) \wedge (x \in C))\} \\ &= \{x : ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))\} \\ &= \{x : (x \in A) \vee (x \in B)\} \cap \{x : (x \in A) \vee (x \in C)\} \\ &= (A \cup B) \cap (A \cup C) \end{aligned}$$

∎

8.20 **Theorem 1.** *Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.*

Proof. Set $A = \{9^n : n \in \mathbb{Q}\}$ and let $B = \{3^n : n \in \mathbb{Q}\}$. We prove the statement in two steps by showing $A \subseteq B$ and $B \subseteq A$. First we show that $A \subseteq B$. Let us assume that $x \in A$. Then there is a rational number n such that

$$x = 9^n = (3^2)^n = 3^{2n}.$$

Since $2n \in \mathbb{Q}$, it follows that $x \in B$.

Now assume that $x \in B$. Then there is a rational number m such that

$$x = 3^m = \left(9^{1/2}\right)^m = 9^{m/2}.$$

Since $m/2 \in \mathbb{Q}$, it follows that $x \in A$. This completes the proof. ∎

8.28 **Theorem.** *Prove that $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.*

[1] You are free to use without proof the distributive rule and all logical equivalences listed on page 52 of Book of Proof, by Richard Hammock.

Proof. We prove the statement in two steps. First, Since 12 and 25 are integers, it is clear that whenever a and b are integers then so is $12a + 25b$. This shows that $\{12a + 25b : a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$.

Next, let $n \in \mathbb{Z}$. Observe that

$$1 = 12(-2) + 25(1),$$

and so

$$n = n \cdot 1 = n\,(12(-2) + 25(1)) = 12(-2n) + 25(n).$$

Since both n and $-2n$ are integers, it follows that $n \in \{12a + 25b : a, b \in \mathbb{Z}\}$. Therefore $\mathbb{Z} \subset \{12a + 25b : a, b \in \mathbb{Z}\}$. This completes the proof. □

Department of Mathematics, Fordham University
Email address: adamski@fordham.edu
URL: www.johnadamski.com