HOMEWORK 9 - DISCRETE MATH SPRING 2023

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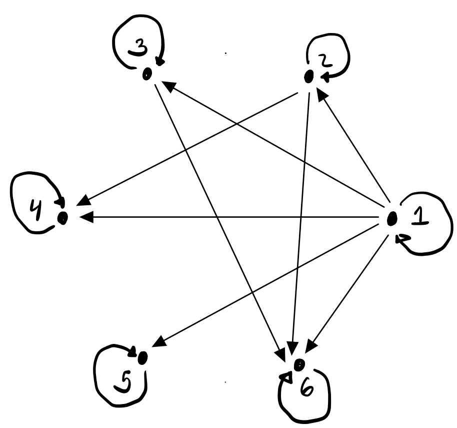
`11.1.2` **Theorem.** *Let $A = \{1, 2, 3, 4, 5, 6\}$. Write out the relation R that expresses $|$ (divides) on A. Then illustrate it with a diagram.*

Proof. By definition,

$$R = \{(x, y) \in A \times A \ : \ x \mid y\}$$
$$= \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,2), (2,4), (2,6), \\ (3,3), (3,6), \\ (4,4), \\ (5,5), \\ (6,6) \end{array} \right\}$$

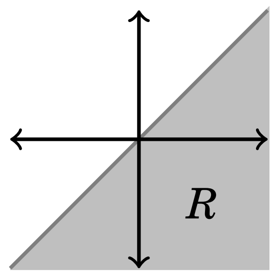


□

`11.1.10` **Theorem.** *Consider the subset $R = (\mathbb{R} \times \mathbb{R}) - \{(x, x) \ : \ x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$. What familiar relation on $\mathbb{R}$ is this? Explain.*

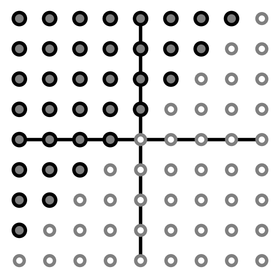
Proof. We see that R contains all ordered pairs of real numbers *except* those with the first and second components/coordinates equal to each other. In other words, xRy if and only if $x \neq y$, i.e. R is the relation $\neq$. □

11.1.12 **Theorem.** *A subset R of $\mathbb{R}^2$ is indicated by gray shading. State this familar relation on $\mathbb{R}$.*



Proof. Since $(x, y) \in R$ if and only if $x \geq y$, we see that R is the relation $\geq$. □

11.1.14 **Theorem.** *A subset R of $\mathbb{Z}^2$ is indicated by gray shading. State this familar relation on $\mathbb{Z}$.*



Proof. Since $(x, y) \in R$ if and only if $x < y$, we see that R is the relation $<$. □

11.2.2 **Theorem.** *Consider the relation $R = \{(a, b), (a, c), (c, c), (b, b), (c, b), (b, c)\}$ on the set $A = \{a, b, c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.*

Proof. The relation R is not reflexive or symmetric, but it is transitive.

- R is not reflexive. In order to be reflexive, R would have to contain $(a, a), (b, b)$ and (c, c). But R does not contain (a, a).
- R is not symmetric. Since R contains (a, b), in order to be symmetric R would have to also contain (b, a), but it does not. The same could be said about $(a, c) \in R$ and $(c, a) \notin R$.
- R is transitive.

□

11.2.6 **Theorem.** *Consider the relation $R = \{(x, x) : x \in \mathbb{Z}\}$. Is this R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?*

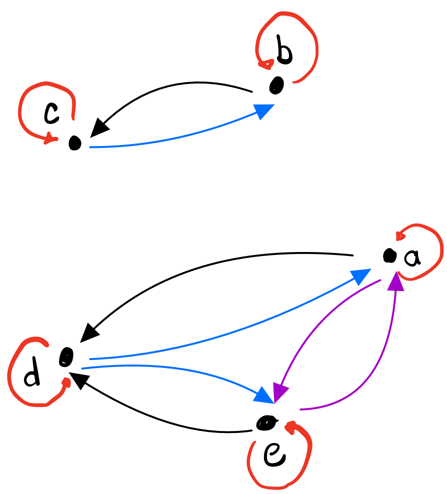
Proof. This relation is reflexive, symmetric, and transitive. It is the familar relation $=$. □

11.2.8 **Theorem.** *Define a relation on $\mathbb{Z}$ as xRy if $|x-y|<1$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?*

Proof. This relation is reflexive, symmetric, and transitive. It is the familar relation $=$. □

11.3.2 **Theorem.** *Let $A=\{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose R has two equivalence classes. Also aRd, bRc, and eRd. Write out R as a set.*

Proof. In the diagram below, the black arrows are the specified relations. The red arrows are necessary in order for R to be reflexive. The blue arrows are necessary in order for R to be symmetric (they point in the opposite direction as the black arrows). The purple arrows are necessary in order for R to be transitive.



Thus,

$$R=\left\{\begin{array}{l}(a,d),(b,c),(e,d),\\(a,a),(b,b),(c,c),(d,d),(e,e),\\(d,a),(c,b),(d,e),\\(a,e),(e,a)\end{array}\right\}$$

□

11.3.10 **Theorem.** *Suppose R and S are two equivalence relations on a set A. Prove that $R\cap S$ is also an equivalence relation.*

Proof. Assume R and S are two equivalence relations on a set A. In order to show that $R \cap S$ is an equivalence relation, we must show that $R \cap S$ is reflexive, symmetric, and transitive.

First we show that R is reflexive. Since both R and S are equivalance relations, both R and S are reflexive. Thus, for any $a \in A$, $(a, a) \in R$ and $(a, a) \in S$. It follows that $(a, a) \in R \cap S$, and so $R \cap S$ is reflexive.

Next we show that $R \cap S$ is symmetric. Since both R and S are equivalence relations, both R and S are symmetric. Assume $(a, b) \in R \cap S$. Then, since $(a, b) \in R$ and R is symmetric, $(b, a) \in R$. Also, since $(a, b) \in S$ and S is symmetric, we have $(b, a) \in S$. Thus, $(b, a) \in R \cap S$, and so $R \cap S$ is symmetric..

Finally we show that $R \cap S$ is transitive. Since both R and S are equivalence relations, both R and S are transitive. Assume $(a, b), (b, c) \in R \cap S$. Then, since $(a, b), (b, c) \in R$ and R is transitive, we have $(a, c) \in R$. Similarly, since $(a, b), (b, c) \in S$ and S is transitive, we have $(a, c) \in S$. Thus, $(a, c) \in R \cap S$, and so $R \cap S$ is transitive. □

`11.4.4` **Theorem.** *Suppose P is a partition of a set A. Define a relation R on A by declaring xRy if and only if $x, y \in X$ for some $X \in P$. Prove R is an equivalence relation on A. Then prove that P is the set of equivalence classes of R.*

Proof. To begin, let us show that R is an equivalence relation on A.

First we show that R is reflexive. Let a be any element of A. Since P is a partition on A, by definition A is the union of all subsets in P. Therefore, a belongs to some $X \in P$. Thus, it is true (though redundant) to say that $a, a \in X$ for some $X \in P$, and so R is relexive.

Next we show that R is symmetric. Assume $a, b \in A$ and aRb. That is, $a, b \in X$ for some $X \in P$. This is clearly equivalent to the statement that $b, a \in X$ for some $X \in P$, and so bRa. This shows that R is symmetric.

Lastly we show that R is transitive. Assume aRb and bRc. Then, by definition, there exists some $X \in P$ such that $a, b \in X$ and some $Y \in P$ such that $b, c \in Y$. Since P is a partition, by definition, if $X \neq Y$ then $X \cap Y = \emptyset$. Equivalently, if $X \cap Y \neq \emptyset$, then $X = Y$ (contrapositive). Since $b \in X \cap Y$, it follows that $X \cap Y \neq \emptyset$, and so $X = Y$. Now we see that a, b, c all belong to the same subset $X \in P$ and, in particular, aRc. This shows that R is transitive and completes the proof that R is an equivalence relation.

To finish the proof, we must show that P is the set of equivalence classes on A. By theorem 11.2 in *Book of Proof* by Richard Hammock, the set $C = \{[a] : a \in A\}$ of equivalence classes of R forms a partition of A. What remains is to show that this partition C is equal to the partition P. We do this using definitions and logical equivalences.

$$\begin{aligned} C &= \{[a] \ : \ a \in A\} \\ &= \{\{x \in A \ : \ xRa\} \ : \ a \in A\} \\ &= \{\{x \in A \ : \ x, a \in X, \text{ for some } X \in P\} \ : \ a \in A\} \\ &= \{\text{the subset in } P \text{ that contains } a \ : \ a \in A\} \\ &= P \end{aligned}$$

□

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