HOMEWORK 11 - DISCRETE MATH SPRING 2023

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`12.4.4` **Theorem.** *Suppose $A = \{a, b, c\}$. Left $f : A \to A$ be the function $f = \{(a, c), (b, c), (c, c)\}$, and let $g : A \to A$ be the function $g = \{(a, a), (b, b), (c, a)\}$. Find $g \circ f$ and $f \circ g$.*

Proof.

$$\begin{aligned} g \circ f &= \{(a, a), (b, a), (c, a)\} \\ f \circ g &= \{(a, c), (b, c), (c, c)\} \end{aligned}$$

□

`12.4.8` **Theorem.** *Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.*

Proof.

$$\begin{aligned} g \circ f(m, n) &= g(3m - 4n, 2m + n) \\ &= \big(5(3m - 4n) + (2m + n), (3m - 4n)\big) \\ &= (17m - 19n, 3m - 4n) \\ f \circ g(m, n) &= f(5m + n, m) \\ &= \big(3(5m + n) - 4m, 2(5m + n) + m\big) \\ &= (11m + 3n, 11m + 2n) \end{aligned}$$

□

`12.4.10` **Theorem 1.** *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defines by the formula $f(x, y) = (xy, x^3)$. Find a formula for $f \circ f$.*

Proof.

$$\begin{aligned} f \circ f(x, y) &= f(xy, x^3) \\ &= \big(xy(x^3), (xy)^3\big) \\ &= (x^4 y, x^3 y^3) \end{aligned}$$

□

`12.5.2` **Theorem.** *In exercise 9 of Section 12.2 you proved that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective. Now find its inverse.*

Proof. Set $y = f(x)$.

$$y = \frac{5x+1}{x-2}$$

Solve for x. This is possible because f is injective.

$$xy - 2y = 5x + 1$$
$$x(y-5) = 2y + 1$$
$$x = \frac{2y+1}{y-5}$$

This is the equation $x = f^{-1}(y)$. It is defined for all y in the codomain of f because f is surjective. Thus

$$f^{-1}(x) = \frac{2x+1}{x-5}.$$

□

12.5.6 **Theorem.** *The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m,n) = (5m + 4n, 4m + 3n)$ is bijective. Find its inverse.*

Proof. Set $(x,y) = f(m,n)$.

$$(x,y) = (5m + 4n, 4m + 3n)$$

This gives a system of 2 equations.

$$5m + 4n = x \tag{1}$$
$$4m + 3n = y \tag{2}$$

Adding -3 times equation (1) and 4 times equation (2), we have

$$m = 4y - 3x.$$

Adding 4 times equation (1) and -5 times equation (2), we have

$$n = 4x - 5y.$$

Thus,

$$(m,n) = (4y - 3x, 4x - 5y), \text{ i.e.}$$
$$f^{-1}(x,y) = (4y - 3x, 4x - 5y).$$

□

12.6.2 **Theorem.** *Consider the function $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$ given as*

$$f = \{(1,3),(2,8),(3,3),(4,1),(5,2),(6,4),(7,6)\}.$$

Find $f(\{1,2,3\})$, $f(\{4,5,6,7\})$, $f(\emptyset)$, $f^{-1}(\{0,5,9\})$, and $f^{-1}(\{0,3,5,9\})$.

Proof.

$$f(\{1,2,3\}) = \{3,8\}$$
$$f(\{4,5,6,7\}) = \{1,2,4,6\}$$
$$f(\emptyset) = \emptyset$$
$$f^{-1}(\{0,5,9\}) = \emptyset$$
$$f^{-1}(\{0,3,5,9\}) = \{1,3\}$$

□

12.6.6 **Theorem.** *Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.*

Proof. No. As a counterexample, consider the function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = 1$ (a constant function), and set $Y = \mathbb{N}$ (the entire codomain). Then $f^{-1}(Y) = \mathbb{N}$ (the inverse of the codomain is always the domain), but $f\big(f^{-1}(Y)\big) = f(\mathbb{N}) = \{1\} \neq Y$.

For this counterexample, notice that the range of f is so small in comparison to the codomain, it is no wonder why we do not recover the original set Y. However, if f *was* surjective then we would always recove the original set Y and the statement $f(f^{-1}(Y)) = Y$ *would* be true. □

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